Note on Dynamic Game with Complete Information

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1 Strategy and Information in Extensive Game

Definition 1.1 (Extensive Form of Game)

1. *Player Set, e.g.* $N = \{1, 2, 3, 4, \cdots, n\}$.
2. *Set of History* H*, terminal history set* Z.
3. *Player function* P*, action set* A_i.
4. *Payoff functions* U_i.

Definition 1.2 (Perfect Information)

A game has perfect information if all information sets are singletons. Otherwise, it has imperfect information.

Definition 1.3 (Pure strategy)

A pure strategy of player i *in an extensive form game with perfect information is a complete list of actions, one action for each decision node that player* i *is entitled to move.*

Example 0.1 For example, Ann has 8 strategies in this game. Liduozhe P4.

Definition 1.4 (Strategy profile)

A strategy profile s *(one strategy* s_i *for each player* i*) determines a sequence of actions leading to a terminal node, namely, a path of play. We refer to this path of play as the outcome of* s.

Lemma 1.1

A finite game of perfect information has a pure strategy Nash equilibrium.

Note on Is PSNE reasonable? *However, some of PSNE are more reasonable than those with incredible threats.*

Proof We use backward induction to solve this game, and from the terminal node to the starting node, we must be able to find a PSNE. ■

Definition 1.5 (Sequential rationality)

A player is sequentially rational iff, at each node he is to move, he maximizes his expected utility conditional on that he is at the node – even if this node is precluded by his own strategy.

Note on Backward induction outcome *In a finite game of perfect information, common knowledge of sequential rationality yields the backward induction outcome.*

Note on Is Backward induction outcome reasonable? *For example, centipede game.*

For example, chain store paradox.

2 Solution Concept 5: Subgame Perfect Equilibrium

Definition 2.1 (Imperfect information)

The simultaneity of moves means that these games have imperfect information.

Note on *We define the subgame-perfect outcome of such games, which is the natural extension of backwards induction to these games. Here the subgame-perfect is different from backward induction outcome since we solve a real game in the 1st step rather than solving a single-person optimization problem.*

Definition 2.2 (Information set)

An information set for a player is a collection of decision nodes in which he has to move.

Note on *In a perfect information game, each information set contains a single decision node.*

Note on *In each information set: a player knows he/she is one of nodes from the information set, however, he/she does not know which node exactly.*

Definition 2.3 (History)

A sequence of decision nodes starting from the initial decision node and connected by actions taken by the players is often referred to as a history.

Note on Terminal history *A terminal history is simply a complete path of play (an outcome), a nonterminal history ends with a decision node where one player is to move.*

Definition 2.4 (Pure strategy in imperfect information)

A pure strategy of player i in an extensive form game is a complete list of actions, one action for each information set that player i is entitled to move.

Note on Number *If player i has K information sets, and at the n^{th} information set, the number of actions is A_n, then the number of pure strategies is*

$$\#S_i = A_1 \times A_2 \times \ldots \times A_K.$$

Definition 2.5 (Equivalent pure strategy)

A player's two pure strategies are equivalent if they lead to the same outcome for every pure strategy profile of other players.

Note on Outcome *Note that outcome means not only payoff, but also the path of play.*

Definition 2.6 (Reduced strategic form)

The reduced strategic form of an extensive game is obtained by eliminating all but one member of each equivalent class.

Note on *Be very careful, this is just a simplification.*

Example 0.1 Example.

Definition 2.7 (Perfect recall)

No players ever forgets any information he once knew, including his past actions.

Note on Perfect recall vs. information *The condition of perfect information is stronger, perfect information means perfect recall, but not vice versa.*

Definition 2.8 (Behavioral strategy)

A behavioral strategy of player i specifies a probability distribution on the set of actions at each information set of player i.

Note on Mixed vs. Behavioral Strategy *A mixed strategy σ_i generates a unique behavioral strategy b_i, and a behavioral strategy b_i can be generated by one or more mixed strategies. σ_i and b_i are equivalent if for any pure strategy profile s_{-i}, (σ_i, s_{-i}) and (b_i, s_{-i}) induce same probability distribution over terminal histories.*

Lemma 2.1

In a game of perfect recall, every mixed strategy is equivalent to the behavioral strategy it generates, and every behavioral strategy is equivalent to each mixed strategy that generates it.

Note on Imperfect recall example

In this game tree, player 1 forgets whether he chooses A or B before. Here player 1's set of pure strategies is $S^1 = \{AC, AD, BC, BD\}$, consider a mixed strategy $\sigma_1 = \left(\frac{1}{2}, 0, 0, \frac{1}{2}\right)$, it can generate the behavioral strategy $b_1 = \left\{\left(\frac{1}{2}, \frac{1}{2}\right), \left(\frac{1}{2}, \frac{1}{2}\right)\right\}$.

Note that b_1 is not equivalent to σ_1. Let player 2 choose L. Then (σ_1, L) induces a probability of 1/2 to path (A,L,C) and (B,L,D) respectively, but (b_1, L) induces a probability of 1/4 to each of the four paths: (A,L,C), (A,L,D), (B,L,C) and (B,L,D).

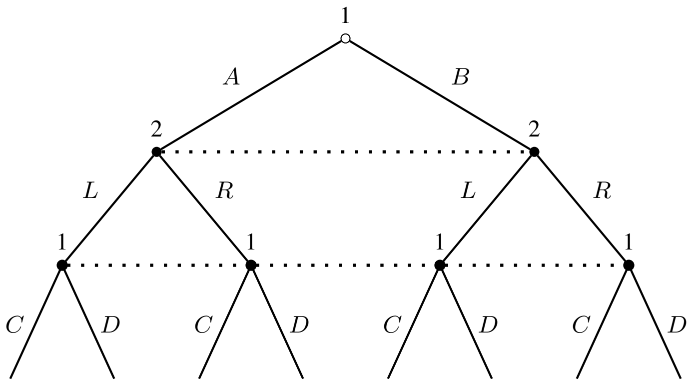

Figure 1: Example of imperfect recall

Definition 2.9 (Subgame)

A subgame is a part of the original game tree with following properties:

- *it begins with an information set containing a single decision node;*
- *it contains all the successor nodes, their information sets, and connecting branches, up to all the relevant terminal nodes.*

Note on *If a subgame contains one node in an information set, it must contain all the nodes in that information set.*

Definition 2.10 (Subgame Perfect Equilibrium (Gibbons, 1992, p. 95))

A strategy profile is a SPE if it induces a NE in every subgame.

Note on *Subgame-perfect Nash equilibrium is a refinement of Nash equilibrium.*

Note on Existence and Uniqueness *Every finite game of perfect information has a pure strategy SPE. Moreover, if no player is indifferent at any two terminal nodes, then there is a unique SPE, which can be derived by backward induction.*

Note on Find SPE

1. *Identify all NEs of the final subgames.*
2. *Select one NE in each final subgame, and replace the subgame with a terminal node with the payoffs of the selected NE.*
3. *Go backwards until a strategy profile of the original game is determined.*

Note on Unreasonable SPE *For example, (Out, Fight) becomes a SPE, which seems unreasonable, Since Fight is not rational for Incumbent. Thus we introduce other solution concepts later.*

Lemma 2.2 (One-step deviation principle)

A strategy profile is subgame perfect iff no player can gain by deviating from the strategy profile in a single information set and conforming to it thereafter.

Note on Nature *The logic here is, if there is no profitable one-step deviation, then there is no profitable (multiple-steps) deviation everywhere.*

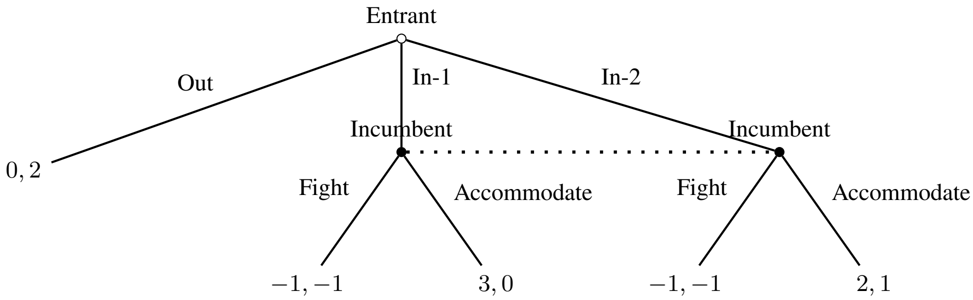

Figure 2: Unreasonable SPE

Example 0.2 Suppose there is no profitable one-step deviation in this game, and the subgame perfect decision is to choose C_1 for player 1 in each period. And we want to show that there is no profitable (multiple-steps) deviation too. The condition actually means $u_1 \geq u_3$, $u_3 \geq u_5$ and $u_5 \geq u_7$, that is, $u_1 \geq u_7$, even though player 1 changes decisions in three periods to C_2 cannot improve his payoff.

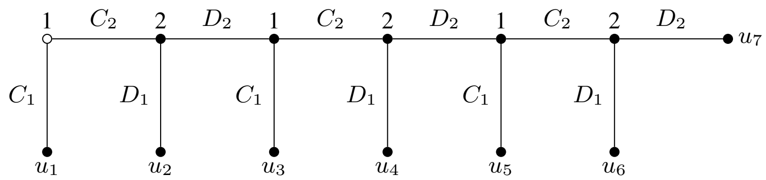

Figure 3: Example of one-step deviation

Theorem 2.1 (Condition for subgame perfect in finite horizon)

In a finite-horizon extensive game, a strategy profile s^* is subgame perfect iff there is no player i and no strategy $\hat{s}_i$ that agrees with s_i^* at all but one but one of player i's information sets, such that $\hat{s}_i$ is a better response to s_{-i}^* than s_i^* conditional on that information set being reached.

Definition 2.11 (Continuous at infinity)

A infinite-horizon extensive game is continuous at infinity if for each player i the payoff function u_i satisfies

$$\sup_{h,\tilde{h} \text{ s.t. } h^t = \tilde{h}^t} \left| u_i(h) - u_i(\tilde{h}) \right| \to 0 \text{ as } t \to \infty$$

where h denotes an infinite-horizon history.

Note on Example *Discounted future payoffs is a example continuous at infinity, consider two history which is the same from 1 to T, and different from $T+1$, e.g. 10 and 100 respectively, the payoff difference between these two history becomes negligible when T is large enough.*

Example 0.3 The following game is not continuous at infinity. The only SPE is to choose C at every decision node, the strategy of choosing D at every node does not have any profitable one-step deviation, but is not subgame perfect; thus, one-step deviation principle fails here.

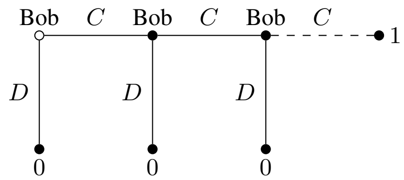

Figure 4: Example of non-continuous at infinity

Theorem 2.2 (Condition for subgame perfect in infinite horizon)

In an infinite-horizon extensive game which is continuous at infinity, a strategy profile s^* is subgame perfect iff there is no player i and no strategy $\hat{s}_i$ that agrees with s_i^* at all but one of player i's information sets, such that $\hat{s}_i$ is a better response to s_{-i}^*, than s_i^* conditional on that information set being reached.

Note on From finite to infinite *With the property of continuous at infinity, suppose there is an infinite-step deviation with profit improvement ε, then you can always find a large T period deviation to get improvement such as $\frac{\varepsilon}{2}$, $\frac{\varepsilon}{3}$ and so on, and finally you can find a one-step profitable deviation.*

3 Examples of finding SPE

3.1 Stackelberg Model (Quantity Competition)

Firm 1 (leader) chooses q_1 first, and then firm 2 chooses q_2 after observing q_1. Say that $p(q) = 100 - q$, and player i's utility is $u_i(q_1, q_2) = [100 - (q_1 + q_2)]q_i$. This game can be solved by backward induction. Given q_1, firm 2's optimal decision is $\frac{100-q_1}{2}$. In the first period, firm 1 foresees firm 2's choice, and his optimal decision is $q_1 = 50$. And the SPE strategy profile is $q_1^* = 50, q_2^* = \frac{100-q_1}{2}$, the SPE outcome is $q_1 = 50, q_2 = 25, p = 25, \pi_1 = 1250, \pi_2 = 625$.

Insights: Information makes palyer 2 worse off, if player 2 cannot see q_1, then this is a simultaneous game (cournot game), and player 2 enjoys higher payoff. This is an example of first-mover advantage.

3.2 Price Competition

Firm 1 is again the leader and sets its price first and firm 2 is the follower setting its price second. Each firm produces an identical good at marginal cost c, and consumers will purchase the good at the lower price. If they set the same prices then each firm will serve half the market. Then the subgame equilibrium is that firm 1 set a price equal to unit cost c and firm 2's best response is to match firm 1's price.

Suppose two firms are not selling identical products. Production differentiation changes the outcome of price competition quite a bit. There is a product spectrum of unit length along which consumers are uniformly distributed, where firm 1 has the address $x = 0$ and firm 2 has location

$x = 1$. Consumers are identical in their reservation price $V >> c$, and the consumer incurs the loss of marginal traveling cost t. The result is interesting. On difference is that prices now are higher. In the simultaneous price game the two firms set the same prices $p_1^* = p_2^* = c + t$, whereas in the sequential game firm 1 sets a price in stage 1 that is greater than $c + t$, and firm 2 responds by setting a slightly lower price, but still higher than $c + t$. A second difference is that the two firms in the sequential price game have different market shares and earn different profits. In the simultaneous price-setting game, each firm served one half the market. In the sequential game, firm 1 serves $3/8$ of the market, whereas firm 2 serves $5/8$ of the market.

Note that unlike the Stackelberg output game, the sequential price game just described presents a clear second mover advantage.

3.3 Ultimatum Rubinstein Bargaining (Felix, 2022, Lec. 11)

One-period The only SPNE is that: the proposer makes an offer $x^* = 0$, and the responder accepts any offer $x \geq 0$. Actually, the latter is a prediction what the responder would do after receiving any offer.

Two-period The SPNE is that

1. Player 1 offers $x_1 = \delta_2$ in period $t = 1$, and accepts any offer $x_2 \geq 0$ in $t = 2$, and
2. Player 2 offers $x_2 = 0$ in period $t = 2$, and accepts any offer $x_1 \geq \delta$ in $t = 1$.

Now go to infinite periods, let us start from three periods. Note that players are impatient: they discount payoffs received in later periods by the factor δ per period, where $0 < \delta < 1$.

1. At the beginning of the first period, player 1 proposes to take a share s_1 of the dollar, leaving $1 - s_1$ for player 2. Player 2 either accepts the offer or rejects the offer.
2. At the beginning of the 2nd period, player 2 proposes that player 1 take a share s_2 of the dollar, leaving $1 - s_2$ for player 2. Player 1 either accept the offer or rejects the offer.
3. Player 1 get s, and player 2 get $1 - s$.

Three-period:

(iii) player 1 and 2 get s and $1 - s$ respectively.

(ii) player 1 accepts the offer only when $s_2 > \delta s$, and player 2 faces a trade-off between $1 - \delta s$ and $\delta(1 - s)$, obviously $1 - \delta s$ is better, thus the optimal decision for player 2 is $s_2^* = \delta s$.

(i) player 2 accepts the offer only when $1 - s_1 \geq \delta\left(1 - s_2^*\right)$ $\left(s_1 \leq 1 - \delta\left(1 - s_2^*\right)\right)$. Player 1 faces a trade-off between $1 - \delta\left(1 - s_2^*\right)$ and δs_2^*, obviously $1 - \delta\left(1 - s_2^*\right)$ is better, thus $s_1^* = 1 - \delta(1 - \delta s)$.

- The backwards outcome is player 1 offers $(s_1^*, 1 - s_1^*)$, and player 2 accepts it.

Infinite-period: Suppose there is a backwards induction outcome where player 1 and 2 get s and $1 - s$ respectively, then we can use the equilibrium result $(f(s), 1 - f(s))$ in the two-period to derive the new backwards-induction outcome, here $f(s) = 1 - \delta(1 - \delta s)$. Let s_H be the highest payoff in these backwards-induction outcome that palyer 1 can achieve, imagine it is player 1's third period profit, and backwards to the first period, player 1's profit is $f(s_H)$. And

$f(s)$ increases in s, thus $f(s_H)$ is the higher one, by assumption we have $f(s_H) = s_H$, and also $f(s_L) = s_L$. The only solution for $f(s) = s$ is $s^* = s_H = s_L \frac{1}{1+\delta}$. That is, this game has a unique backwards-induction outcome, where player 1 offers $(s_1^*, 1 - s_1^*)$ in the first period, and player 2 accepts it.

Interpretation: In bargaining games, patience works as a measure of bargaining power: To be continue

Multilateral bargaining

3.4 Strategy Pre-commitment (Felix, 2022, Lec. 10)

Note on Example: Advertising and Competition

Note on Example: Entry-deterrence game

Definition 3.1 (Top dog, puppy dog ploy, lean and hungry look, fat cat strategy)

3.5 Imperfect information: Tournament

1. The boss choose wage w_H and w_L to maximize payoff $y_1 + y_2 - w_H - w_L$.
2. Two worker chooses effort e_i to maximizes their payoff $u(w, e) = w - g(e)$, where $g(e)$ is increasing and convex (i.e., $g'(e) > 0, g''(e) > 0$).
3. Worker's output realize as $y_i = e_i + \varepsilon_i$, where ε_i is noise and iid to $f(\varepsilon)$ with zero mean. The winner earns w_H while the loser earns w_L.

Here we ignore the possibilities of asymmetric equilibria and an equilibrium given by the corner solution $e_1 = e_2 = 0$.

Second period: Suppose w_H and w_L are given, then NE (e_1^*, e_2^*) should satisfy the following conditions:

$$\begin{aligned}
&\max_{e_i \geq 0} \quad w_H \operatorname{Prob}\left\{y_i\left(e_i\right)>y_j\left(e_j^*\right)\right\}+w_L \operatorname{Prob}\left\{y_i\left(e_i\right) \leq y_j\left(e_j^*\right)\right\}-g\left(e_i\right) \\
&\quad=\left(w_H-w_L\right) \text{ Prob } \left\{y_i\left(e_i\right)>y_j\left(e_j^*\right)\right\}+w_L-g\left(e_i\right)
\end{aligned}$$

where

$$\begin{aligned}
\operatorname{Prob}\left\{y_i\left(e_i\right)>y_j\left(e_j^*\right)\right\} &= \operatorname{Prob}\left\{\varepsilon_i>e_j^*+\varepsilon_j-e_i\right\} \\
&= \int_{\varepsilon_j} \operatorname{Prob}\left\{\varepsilon_i>e_j^*+\varepsilon_j-e_i \mid \varepsilon_j\right\} f\left(\varepsilon_j\right) d \varepsilon_j \\
&= \int_{\varepsilon_j}\left[1-F\left(e_j^*-e_i+\varepsilon_j\right)\right] f\left(\varepsilon_j\right) d \varepsilon_j
\end{aligned}$$

and the FOC is

$$
\begin{aligned}
& (w_H - w_L)\frac{\partial \operatorname{Prob}\left\{y_i(e_i) > y_j\left(e_j^*\right)\right\}}{\partial e_i} = g'(e_i) \\
\Rightarrow & (w_H - w_L)\int_{\varepsilon_j} f\left(e_i^* - e_i + \varepsilon_j\right) f\left(\varepsilon_j\right) d\varepsilon_j = g'(e_i) \\
\Rightarrow & (w_H - w_L)\int_{\varepsilon_j} f\left(\varepsilon_j\right)^2 d\varepsilon_j = g'(e^*) \quad (SymmetricNashequilibrium)
\end{aligned}
$$

There are two insights: (i) a bigger prize $(w_H - w_L)$for winning induces more effort, (ii) it is not worthwhile to work hard when output is very noisy, because the outcome of the tournament is likely to be determined by luck rather than effort.

Suppose worker's alternative employment opportunity is U_a, then there is a constraint $\frac{1}{2}w_H + \frac{1}{2}w_L - g(e^*) \geq U_a$, and in optimality we have bounded $w_L = 2U_a + 2g(e^*) - w_H$. And the boss's objective is to maximize $2e^* - w_H - w_L$, replace it we have $2e^* - 2U_a - 2g(e^*)$, which is equivalent to choose e^* to maximize $e^* - g(e^*)$, where $g'(e^*) = 1$. That is, the optimal decision should satisfy

$$
(w_H - w_L)\int_{\varepsilon_j} f\left(\varepsilon_j\right)^2 d\varepsilon_j = 1
$$

3.6 Secret Price Cut and Imperfect monitoring (Abreu et al., 1986)

4 Open-loop and Closed-loop Equilibria

Definition 4.1 (Open-loop, closed-loop (Fudenberg and Levine, 1988))

In the open-loop model, players cannot observe the play of their opponents; in the closed-loop model, all past play is common knowledge at the beginning of each stage.

Note on Nature *Open-loop and closed-loop equilibria are then the perfect equilibria corresponding to the two information structures.*

Note on Open vs. Closed *Open-loop equilibria are more tractable than closed-loop equilibria, because players need not consider how their opponents would react to deviations from the equilibrium path. This is why sometimes economists prefer the open-loop, even though the closed-loop is more practical.*

5 Repeated Games

5.1 From Finite Period to Infinite Period

Proposition 5.1 (Finitely repeated game's equilibrium (Gibbons, 1992, p. 84))

If the stage game G has a unique Nash equilibrium then, for any finite T, the repeated game $G(T)$ has a unique subgame-perfect outcome: the Nash equilibrium of G is played in every stage (Backward Induction).

Note on Multiple equilibria case *In a stage game with multiple equilibria, its SPE may contains other outcomes (i.e., non-NE) in periods $t < T$. For example, the stage game has two NEs: (C,M), (B,R). In any SPE, the outcome at the last period must be either (C,M) OR (B,R). Suppose $T = 2$, there is a SPE, where in period 1, player 1 (2) chooses T (L), and in period 2, player 1 (2) chooses C (M) if the outcome of the first period is (T,L), chooses B (R) otherwise. Since for each player, payoffs from ((T,L), (C,M)) is better than payoffs from ((C,L), (B,R)), i.e., $8 + 4 > 9 + 1$.*

	L	M	R
T	$8, 8$	$0, 9$	$0, 0$
C	$9, 0$	$4, 4$	$0, 0$
B	$0, 0$	$0, 0$	$1, 1$

The nature of this phenomenon is the threat is severe enough to deter other's deviation. Follow the similar track, we can incorporate (T,M) into SPE in a three-period game. This phenomenon becomes more obvious in infinitely repeated game, even though there is a unique NE in each period, SPE may contains other outcome (i.e., non-NE) in periods.

Definition 5.1 (Feasible payoffs (Gibbons, 1992, p. 96))

We call the payoffs $(x_1, ..., x_n)$ feasible in the stage game G if they are a convex combination of the pure-strategy payoffs of G.

Definition 5.2 (Average payoff (Gibbons, 1992, p. 97))

Given the discount factor δ, the average payoff of the infinite sequence of payoffs $\pi_1, \pi_2, \pi_3...$ is $C = (1 - \delta) \sum_{t=1}^{\infty} \delta^{t-1} \pi_t$.

Note on Normalization *By this definition, we can treat the sequence $\pi_1, \pi_2, \pi_3...$ as a constant sequence $C, C, C....$*

Definition 5.3 (History and Outcome path)

A history (up to period t) h_t is a sequence of past observed outcomes in the stage game, i.e., $h_t = (a^0, a^1, \ldots, a^{t-1})$. Initial history is written as h_0. An outcome path $\mathbf{a} = (a^0, a^1, \ldots)$ is an infinite history.

Definition 5.4 (Pure strategy and Behavioral strategy)

A pure strategy s_i of player i specifies an action $s_i(h_t) \in A_i$ for every h_t. A behavioral strategy σ_i specifies a randomization over A_i for every h_t, i.e., $\sigma_i(h_t) \in \Delta(A_i)$.

Lemma 5.1 (One-step deviation principle)

A strategy profile σ is a SPE of $G^{\infty}(\delta)$ iff it passes the following test: after any history h_t, every player i, assuming that all but himself will play according to σ at t, and that all (including himself) will follow σ at $t+1$ and thereafter, does not have an incentive to deviate from $\sigma_i(h_t)$ at t.

Theorem 5.1 (Infinitely repeated game's equilibrium (Gibbons, 1992, p. 97))

Let G be a finite, static game of complete information. Let $(e_1, ..., e_n)$ denote the payoffs from a Nash equilibrium of G, and let $(x_1, ..., x_n)$ denote any other feasible payoffs from G. If $x_i > e_i$ for every player i and if δ is sufficiently close to one, then there exists a subgame-perfect Nash equilibrium of the infinitely repeated game $G(\infty, \delta)$ that achieves $(x_1, ..., x_n)$ as the average payoff.

Note on Motivation *The motivation to model an infinitely repeated game is, though no one lives forever in reality, but people don't know when the game ends, in other words, the time period is stochastic.*

5.2 Strategy in Infinite Period

Next we show some strategies by the infinitely repeated prisoners' dilemma. First of all, both players choose D regardless of the history is a SPE.

	C	D
C	$3,3$	$0,4$
D	$4,0$	$1,1$

Definition 5.5 (Grim Trigger Strategy)

Choose C in the first period, stay with C if the opponent has never defected, otherwise switch to D forever.

Note on NE? *When δ is sufficiently large (close to 1), the strategy pair (grim trigger, grim trigger) is a NE. The payoff stream for this strategy pair is (3,3,...), and the discounted average payoff is 3; if he deviates to D, the payoff stream will become (4,1,1...), the discounted average payoff is $4-3\delta$, and $3 \geq 4-3\delta$ when $\delta \geq \frac{1}{3}$.*

$$(1-\delta)\left(4+\frac{\delta}{1-\delta}\right)=4-3\delta$$

Note on SPE? *This strategy pair is not a SPE. Consider the subgame following the outcome (C,D) in the first period:*

1. *Suppose that player 1 adheres to the grim trigger and chooses D in the second period. It is optimal for player 2 to switch to D too, which is not consistent with the grim trigger.*
2. *Suppose that player 2 adheres to the grim trigger and chooses C in the second period. When $\delta \geq \frac{1}{3}$, it is optimal for player 1 not to switch to D.*

Definition 5.6 (Tit-for-tat strategy)

Choose C at $t = 1$, then do whatever the other player did in the previous period.

Note on NE? *The strategy pair (tit-for-tat, tit-for-tat) is a NE when $\delta \geq \frac{1}{3}$. The payoff stream for this strategy pair is (3,3,...), and the discounted average payoff is 3; if a player deviates to D in period t, then he may either alternate between D and C, or chooses D at every period.*

1. *Alternate between D and C: his payoff stream is (4,0,4,0...), the discounted average payoff is*
$$(1-\delta)\frac{4}{(1-\delta^2)} = 4/(1+\delta).$$
2. *Chooses D at every period: his payoff stream is (4,1,1...), the discounted average payoff is*
$$(1-\delta)\left(4+\frac{\delta}{1-\delta}\right) = 4-3\delta.$$

Note on SPE? *The strategy pair (tit-for-tat, tit-for-tat) is a SPE only when $\delta = \frac{1}{3}$.*

1. *History ending in (C,D):*
2. *History ending in (D,C):*
3. *History ending in (D,D):*

Definition 5.7 (Modified grim trigger strategy)

Start with C, and stay with C iff both have been choosing C before (in other words, switch to D forever iff someone, including himself, has defected before).

Note on SPE *When δ is sufficiently large, i.e. $\delta \geq \frac{1}{3}$, the pair of modified grim trigger strategy is a SPE. To see this,*

Note on Another example *It would be better to rewrite this section and supplemtns the visualization.(Felix, 2022, Lec. 12)*

Definition 5.8 (Limited punishment)

Start with C, switch to D for k periods whenever someone (including himself) defects, and then switch back to C.

Note on SPE *When δ and k are sufficiently large, the strategy pair in which each player uses the k-period punishment strategy is a SPE. To be continue,*

Note on *In the infinitely repeated PD, cooperation can be sustained in a SPE when players are sufficiently patient.*

5.3 Folk Theorem

Theorem 5.2 (Folk theorem with Trigger strategy)

If a^* is a NE of the stage game of an infinitely repeated game, then any action profile Pareto dominates (improves all players' payoff) is a SPE by using a^*-trigger strategy if discount factor is sufficiently large.

Remark In the prisoner's dilemma example, (D,D) is an NE but (C,C) Pareto dominates (D,D) so we can use (D,D) to support (C,C) when players are sufficiently patient (δ_i are large).

Definition 5.9 (Minmax payoff and action profile)

The minmax payoff of player i:

$$\underline{v}_i = \min_{\alpha_{-i} \in \Pi_{j \neq i} \Delta(A_j)} \max_{a_i \in A_i} u_i\left(a_i, \alpha_{-i}\right).$$

The minmax action profile for $i4$:

$$m^i = \left(m_1^i, \ldots, m_n^i\right)$$

Note on *The minmax payoff of your opponent actually denotes the worst punishment you can impose to it.*

Note on Example

Definition 5.10 (Individually rational payoff and profile)

1. *A payoff profile v is (strictly) individually rational if $(v_i > \underline{v}_i)$ $v_i \geq \underline{v}_i$ for all i.*
2. *An action profile a is (strictly) individually rational if $u(a)$ is (strictly) individually rational.*

Note on Example

Lemma 5.2

Every Nash equilibrium payoff profile of the stage game G is individually rational.

Theorem 5.3 (Folk theorem 1(Fudenberg and Maskin 1986))

Let V be the set of feasible payoff profiles. Assume either

1. *dim $V \geq N - 1$ or*
2. *projection of V to any two player is two dimensional.*

Then, for any strictly individually rational and feasible v, there is $\bar{\delta} < 1$ such that for any $\delta \in (\bar{\delta}, 1)$, there is a SPE s of $G^\infty(\delta)$ with $U(s) = v$.

Theorem 5.4 (Folk theorem 2)

Let a^* be a strictly individually rational action profile of G. Assume that minmax strategy profile m^i is pure for each i. Assume that there is a collection $\left(a^i\right)_{i \in N}$ of strictly individually rational action profiles of G such that for every $i \in N$, $u_i\left(a^*\right) > u_i\left(a^i\right)$ and

$u_i\left(a^j\right) > u_i\left(a^i\right)$ for $j \neq i$. Then there is $\bar{\delta} < 1$ such that for every $\delta \in (\bar{\delta}, 1)$, there is a SPE of $G^\infty(\delta)$ in which a^* is played on the equilibrium path.

Proof

■

Note on "mutual minimaxing" method in two-player game *For two-player games, we can adopt a simpler SPE strategy profile: after a deviation by either player, each player minimaxes the other for a certain number of periods, after which they return to the original path; if a further deviation occurs during the punishment phase, the phase is begun again.*

However, this method of "mutual minimaxing" does not extend to games with three or more players. This is because with three players, there may not exist an action profile in whichevery player is minimaxed by the other two. This is why we need the dimensionality condition in the folk theorem. For example, p59.

Note on Interpretation *Actually, folk theorem shows that any point on the edge or interior of the feasible individually rational region can be supported as a SPNE of the infinitely-repeated game as long as the discount factor δ is close enough to 1, i.e., players care about the future.*

Note on Partial cooperation and example

Note on Advantaegs and disadvantages

5.4 Examples of Repeated Game

5.4.1 Repeated Cournot Game

Original cournot game: Two firms, market clearing price $P(Q) = a - q_1 - q_2$, each firms has a marginal cost c and no fixed costs, there is a unique equilibrium $q_{NE} = \frac{a-c}{3}$ and $p_{NE} = \frac{a+2c}{3}$. And in the monopoly case, we have $q_M = \frac{a-c}{2}$ and $p_M = \frac{a+c}{2}$.

Define a trigger strategy: each firm produces $q_M/2$, if each of them has done so in all previous periods; otherwise produce q_{NE}. Easy to see profit under cooperation is $\pi_C = \pi_M/2 = \frac{(a-c)^2}{8}$, and the profit under punishment is $\pi_{NE} = \frac{(a-c)^2}{9}$. Suppose one firm intends to deviates from cooperation, his profit will be $\pi_D = \max_q \left(a - q - \frac{q_M}{2} - c\right) q = \frac{9(a-c)^2}{64}$.

- Payoff of cooperation: $\sum_{t=0}^{\infty} \delta^t \pi_C = \frac{1}{1-\delta}\pi_C$.
- Payoff of deviation: $\pi_D + \sum_{t=1}^{\infty} \delta^t \pi_{NE} = \pi_D + \frac{\delta}{1-\delta}\pi_{NE}$.

The condition for cooperation in NE is $\frac{1}{1-\delta}\pi_C \geq \pi_D + \frac{\delta}{1-\delta}\pi_{NE}$, that is, $\delta \geq \frac{9}{17}$. Thus only when the discount factor is large enough, will firms keep cooperation in the long run. The question is, what happen when $\delta < \frac{9}{17}$?

Suppose now the trigger strategy is: each firm produces q_C, if each of them has done so in all previous periods; otherwise produce q_{NE}. Easy to see profit under cooperation is $\pi_C = (a - 2q_C - c)\, q_C$, and the profit under punishment is $\pi_{NE} = \frac{(a-c)^2}{9}$. Suppose one firm intends to deviates from cooperation, his profit will be $\pi_D = \max_q (a - q - q_C - c)\, q = \frac{(a-q_C-c)^2}{4}$.

The condition for cooperation in NE is $\frac{1}{1-\delta}\pi_C \geq \pi_D + \frac{\delta}{1-\delta}\pi_{NE}$, and by finding the largest q_C we have $q_C = \frac{9-5\delta}{3(9-\delta)}(a-c)$. Note that $\frac{dq_C}{d\delta} < 0$, when $\delta \to 9/17$ we have $q_C \to q_M/2$, when $\delta \to 0$ we have $q_C \to q_{NE}$.

5.4.2 Efficiency Wages

1. The firm offers the worker a wage w.
2. The worker accepts or rejects the firm's offer with outside wage w_0. If accept, the worker chooses either to supply effort (which entails disutility e) or to shirk. Note that the effort decision is not observed by the firm. Output can either high $y > 0$ or low 0. Here with effort the output is sure to be high, otherwise it is high with probability p and low with probability $1-p$.

5.4.3 Time-Consistent Monetary Policy

1. Employers form an expectation of inflation π^e.
2. The monetary authority observes this expectation and chooses actual inflation pi. The payoff to employers is $-(\pi - \pi^e)^2$, and employers simply want to anticipate inflation correctly, they achieve their maximum payoff (zero) when $\pi = \pi^e$. The monetary authority would like inflation to be zero but output y to be at its efficient level y^*, its payoff is $U(\pi, y) = -c\pi^2 - (y - y^*)^2$, where $c > 0$ reflects the tradeoff between its two goals. And the auctual output is $y = by^* + d(\pi - \pi^e)$, where $b < 1$ reflects the presence of monopoly power in product markets and $d > 0$ measures the effect of surprise inflation on output.

5.4.4 Discrete Cournot Game (Penal code) (Abreu, 1988)

5.4.5 Fluctuating Demand and Perfect monitoring (Rotemberg and Saloner, 1986)

5.4.6 Quid Pro Quo

Definition 5.11 (Quid Pro Quo)

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